Let $f \in S(\mathbb{R})$. Recall

- Fourier Inversion Formula:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} d\xi$$

- Plancherel Formula:

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi$$

- Poisson Summation Formula:

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}$$

I. Show that if $f \in S(\mathbb{R})$ and $\hat{f}(\xi) = 0$ for $|\xi| \geq \frac{1}{2}$,

then (a) $f(x) = \sum_{n=-\infty}^{\infty} f(n) K(x-n)$ where $K(y) = \dfrac{\sin \pi y}{\pi y}$

(b) $\int_{-\infty}^{\infty} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |f(n)|^2$

Proof of (a):

By Fourier Inversion Formula,

$$f(-x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi (-x)} d\xi$$

$$= \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-2\pi i \xi x} d\xi$$

$$= \hat{\hat{f}}(x).$$

Applying Poisson Summation Formula to $\hat{f}$, we have

$$\sum_{n=-\infty}^{\infty} \hat{f}(\xi+n) = \sum_{n=-\infty}^{\infty} \hat{\hat{f}}(n) e^{2\pi i n \xi}$$

$$= \sum_{n=-\infty}^{\infty} f(-n) e^{2\pi i n \xi}$$

$$= \sum_{n=-\infty}^{\infty} f(n) e^{-2\pi i n \xi}$$

For any $\xi \in (-\frac{1}{2}, \frac{1}{2})$ and $n \in \mathbb{Z} \setminus \{0\}$, $|\xi + n| > \frac{1}{2}$. Thus for any $\xi \in (-\frac{1}{2}, \frac{1}{2})$

$$\hat{f}(\xi) = \sum_{n=-\infty}^{\infty} \hat{f}(\xi+n) = \sum_{n=-\infty}^{\infty} f(n) e^{-2\pi i n \xi}.$$

By Fourier Inverison Formula,

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{n=-\infty}^{\infty} f(n) e^{-2\pi i n \xi} \right) e^{2\pi i x \xi} d\xi$$

$$= \sum_{n=-\infty}^{\infty} f(n) \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i (x-n)\xi} d\xi$$

$$= \sum_{n=-\infty}^{\infty} f(n) \left(\frac{e^{\pi i (x-n)} - e^{-\pi i (x-n)}}{2\pi i (x-n)} \right)$$

$$= \sum_{n=-\infty}^{\infty} f(n) \frac{\sin(\pi(x-n))}{\pi(x-n)}$$

$$= \sum_{n=-\infty}^{\infty} f(n) K(x-n)$$

$\square$

Proof of (b):

Recall for any $x \in (-\frac{1}{2}, \frac{1}{2})$,

$$\hat{f}(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n+x) = \sum_{n=-\infty}^{\infty} f(n) e^{-2\pi i n x}.$$

Define a 1-periodic function g by

$$g(x) = \hat{f}(x) \quad \text{for } x \in (-\tfrac{1}{2}, \tfrac{1}{2})$$

Then the Fourier coefficients of g is given by

$$\hat{g}(-n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} g(x) e^{+2\pi i n x} dx$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{f}(x) e^{+2\pi i n x} dx$$

$$= \int_{-\infty}^{\infty} \hat{f}(x) e^{+2\pi i n x} dx$$

$$= f(n)$$ Fourier Inversion

Thus $\displaystyle\sum_{n=-\infty}^{\infty}|f(n)|^2 = \sum_{n=-\infty}^{\infty}|\hat{g}(-n)|^2$

$$= \sum_{n=-\infty}^{\infty}|\hat{g}(n)|^2$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}}|g(x)|^2\,dx \qquad \text{Parseval Identity}$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}}|\hat{f}(x)|^2\,dx$$

$$= \int_{-\infty}^{\infty}|\hat{f}(x)|^2\,dx$$

$$= \int_{-\infty}^{\infty}|f(x)|^2\,dx. \qquad \text{Plancherel Formula}$$

$\square$

II. Calculate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(1+x^2)(4+x^2)}\,dx$.

Proof: As in the midterm, we give the general form of Plancherel Formula:

$$\int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\,dx = \int_{-\infty}^{\infty} \hat{f}(\xi)\,\overline{\hat{g}(\xi)}\,d\xi$$

Recall $e^{-|x|} \xrightarrow{\tilde{\mathcal{F}}} \dfrac{2}{1+4\pi^2\xi^2}$

Then $e^{-2\pi|x|} \xrightarrow{\tilde{\mathcal{F}}} \dfrac{1}{2\pi}\dfrac{2}{1+\xi^2}$

$\pi e^{-2\pi|x|} \xrightarrow{\tilde{\mathcal{F}}} \dfrac{1}{1+\xi^2}$

Similarly $e^{-4\pi|x|} \xrightarrow{\tilde{\mathcal{F}}} \dfrac{1}{4\pi}\dfrac{2}{1+\frac{\xi^2}{4}}$

$\frac{\pi}{2} e^{-4\pi|x|} \xrightarrow{\tilde{\mathcal{F}}} \dfrac{1}{4+\xi^2}$

By Plancherel Formula,

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)(4+x^2)}\,dx = \frac{\pi^2}{2}\int_{-\infty}^{\infty} e^{-6\pi|x|}\,dx$$

$$= \pi^2 \int_0^{\infty} e^{-6\pi x}\,dx$$

$$= \pi^2 \frac{1}{-6\pi}(-1)$$

$$= \frac{\pi}{6}$$

$\square$

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \\ \lim_{t \to 0} u(x,t) = 0 \end{cases}$$

Clearly, $u(x,t) = 0$ is a solution.

$u(x,t) := \frac{x}{t} \mathcal{H}_t(x)$ where $\mathcal{H}_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$ is the heat kernel.

1 (a) u satisfies the heat equation.

$$\frac{\partial u}{\partial t} = -\frac{x}{t^2}\mathcal{H}_t(x) + \frac{x}{t}\left(\frac{\partial}{\partial t}\mathcal{H}_t(x)\right)$$

$$= -\frac{x}{t^2}\mathcal{H}_t(x) + \frac{x}{t}\left(-\frac{1}{2t}\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}} + \frac{x^2}{4t^2}\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}\right)$$

$$= \mathcal{H}_t(x)\left(-\frac{x}{t^2} - \frac{x}{2t^2} + \frac{x^3}{4t^3}\right).$$

$$\frac{\partial u}{\partial x} = \frac{1}{t}\mathcal{H}_t(x) + \frac{x}{t}\frac{\partial}{\partial x}(\mathcal{H}_t(x))$$

$$= \frac{1}{t}\mathcal{H}_t(x) + \frac{x}{t}\left(-\frac{x}{2t}\mathcal{H}_t(x)\right)$$

$$= \mathcal{H}_t(x)\left(\frac{1}{t} - \frac{x^2}{2t^2}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\mathcal{H}_t(x)\left(\frac{1}{t} - \frac{x^2}{2t}\right) + \mathcal{H}_t(x)\left(-\frac{x}{t^2}\right)$$

$$= \mathcal{H}_t(x)\left(-\frac{x}{2t}\right)\left(\frac{1}{t} - \frac{x^2}{2t^2}\right) + \mathcal{H}_t(x)\left(-\frac{x}{t^2}\right)$$

$$= \mathcal{H}_t(x)\left(-\frac{x}{2t^2} - \frac{x^3}{4t^3} - \frac{x}{t^2}\right)$$

$$= \frac{\partial u}{\partial t}.$$

(b) $\lim\limits_{t\to 0^+} u(x,t) = 0$ for any $x \in \mathbb{R}$.

Pf: $\lim\limits_{t\to 0^+} u(x,t) = \lim\limits_{t\to 0^+} \frac{x}{t}\frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$

$(y = \frac{1}{t})$ $\quad = \lim\limits_{y\to\infty} xy \frac{\sqrt{y}}{\sqrt{4\pi}} e^{-\frac{x^2}{4}y}$

$= \lim\limits_{y\to\infty} \frac{x}{\sqrt{4\pi}} y^{\frac{3}{2}} e^{-\frac{x^2}{4}y}$

When $x=0$, $\frac{x}{\sqrt{4\pi}} y^{\frac{3}{2}} e^{-\frac{x^2}{4}y} = 0$, $\forall\, y>0$.

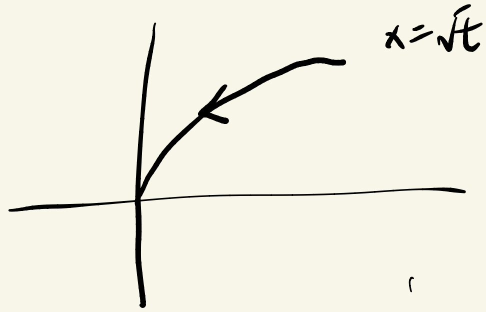
When $x \neq 0$, $e^{\frac{x^2}{4}y}$ grows faster than $y^{\frac{3}{2}}$.

Then $\lim_{y\to\infty} \frac{x}{\sqrt{4\pi}} y^{\frac{3}{2}} e^{-\frac{x^2}{4}y} = 0$

Hence $\lim_{t\to 0^+} u(x,t) = 0$.

(c) u is not continuous at $(0,0)$.

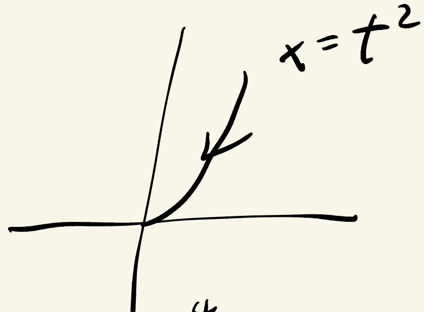
Proof: • We put $x=\sqrt{t}$ and choose the path $(\sqrt{t}, t) \to (0,0)$



$$\lim_{(\sqrt{t},t)\to(0,0)} u(x,t) = \lim_{t\to 0^+} \frac{1}{\sqrt{t}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{1}{4}} = \infty .$$

• On the other hand,

we put $x=t^2$ and choose the path $(t^2, t) \to (0,0)$



$$\lim_{(t^2,t)\to(0,0)} u(x,t) = \lim_{t\to 0} \frac{t^2}{t} \frac{1}{\sqrt{4\pi t}} e^{-\frac{t^4}{4t}}$$